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ON THE UNSTEADY MASS TRANSFER ON A DROP IN A VISCOUS FLUID STREAM

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An analytic solution of the axisymmetric problem of unsteady convective diffusion at the surface of an absorbing drop is obtained for a wide class of unsteady flows of a viscous incompressible fluid. Explicit formulas for the dependence of diffusion fluxes on time are derived in the case of steady translational and purely shear flow around at low Reynolds numbers and unsteady diffusion. The asymptotics of solution of the problem of diffusion on a bubble in a uniformly accelerating stream are obtained for short times.

1. Unsteady diffusion on the surface of an absorbing drop. The general solution. Let us consider the process of unsteady convective diffusion on the surface of a spherical drop in a viscous fluid stream, on the assumption of a high Péclet number. On the assumption of total absorption of the substance dissolved in the stream on the drop surface and constant concentration away from it, using the boundary layer approximation, we write the dimensionless equation of convective diffusion as

$$\frac{\partial c}{\partial t}+\frac{(1+y)^{-2}}{\sin\theta}\frac{\partial(c,\psi)}{\partial(y,\theta)}=P^{-1}\frac{\partial^2 c}{\partial y^2}\quad\left(P=\frac{aU}{D}\right)\tag{1.1}$$

$$c(t,y,\theta^-)=1,\quad c(t,0,\theta)=0,\quad c(t,\infty,\theta)=1$$

where $r=y+1$, θ is the spherical system of coordinates attached to the drop center with angle θ measured from the efflux trajectory θ^+ (from the direction of flow at infinity in the case of translational flow past the drop); c is the concentration; D is the coefficient of diffusion; ψ is the stream function; $\partial(c,\psi)/\partial(y,\theta)$ is the Jacobian of functions c and ψ; the measurement units are: the drop radius a, the characteristic velocity (at infinity) U, and time a/U.

The first boundary condition in (1.1) corresponds to the usual condition of flow-on (the singular streamline $\theta^-(\theta^+)$ on which the normal velocity component near the particle is directed toward (away from) its surface is called the flow-on (flow-off) trajectory [1]).

We assume that the stream function near the drop (bubble) surface may be represented in the form

$$y\to 0,\quad \psi(t,y,\theta)\to\Omega(t)\,yf(\theta);\quad \Omega(t)\geqslant 0\tag{1.2}$$

As will be shown in Sect. 3, the representation (1.2) is valid for translational and shear flows past a bubble.

To simplify the analysis we consider below the region $\sigma=\{\theta^+\leqslant\theta\leqslant\theta^-\}$ where $f(\theta)\geqslant 0$ and the flow-on θ^- and flow-off θ^+ trajectories are defined as follows:

$$f(\theta^-) = f(\theta^+) = 0, \quad -\infty < [f_\theta' / \sin\theta]_{\theta=\theta^-} < 0 \qquad (1.3)$$
$$0 < [f_\theta' / \sin\theta]_{\theta=\theta^+} < \infty$$

The more general case of an arbitrary number of critical lines on the drop surface can be considered in a similar manner [1].

We seek particular solutions of Eq. (1.1) without specifying initial conditions, and introduce the new variables

$$\eta = P^{1/2} y f(\theta), \quad \zeta = \zeta\ (t,\ \theta) \qquad (1.4)$$

assuming that the concentration $c = c\ (\eta,\ \zeta)$ depends only on these. Equation (1.1) then assumes the form

$$\left(\zeta_t' - \frac{\Omega(t) f(\theta)}{\sin\theta} \zeta_\theta'\right) \frac{\partial c}{\partial \zeta} = f^2(\theta) \frac{\partial^2 c}{\partial \eta^2} \qquad (1.5)$$

If function $\zeta = \zeta\ (t,\ \theta)$ is taken as the solution of the equation

$$\frac{\partial \zeta}{\partial t} - \frac{\Omega(t) f(\theta)}{\sin\theta} \frac{\partial \zeta}{\partial \theta} = f^2(\theta) \qquad (1.6)$$

the problem of convective diffusion reduces to the conventional equation ot heat conduction

$$\partial c / \partial \zeta = \partial^2 c / \partial \eta^2 \qquad (1.7)$$

We now determine $\zeta = \zeta\ (t,\ \theta)$ from Eq. (1.6) which is equivalent to the following system of ordinary differential equations:

$$\frac{dt}{1} = -\frac{\sin\theta\, d\theta}{\Omega(t) f(\theta)} = \frac{d\zeta}{f^2(\theta)} \qquad (1.8)$$

Integrating the first and the last two of Eqs. (1.8) we obtain

$$G(t, \theta) = \int_0^t \Omega(\xi)\, d\xi + x(\theta) = C_1 \qquad (1.9)$$
$$\zeta = -\int_{\theta_2}^{\theta} f(\xi) \sin\xi\, \Omega^{-1} [t(\xi, C_1)]\, d\xi + C_2, \quad x(\theta) = \int_{\theta_1}^{\theta} \sin\xi f^{-1}(\xi)\, d\xi$$

where θ_1 and θ_2 are some angles of which θ_1 does not determine critical point of the drop surface $(f\ (\theta_1) \neq 0)$. and function $t\ (\theta,\ C_1)$ is obtained by solving the first of Eqs. (1.9) for t, i.e. $G\,[t\ (\theta,\ C_1),\ \theta] \equiv C_1$. From this we obtain the general solution of Eq. (1.6) is (F is an arbitrary function)

$$\zeta = -\int_{\theta_2}^{\theta} f(\xi) \sin\xi\, \Omega^{-1} [t(\xi, C_1)]\, d\xi + F\,[G(t, \theta)] \qquad (1.10)$$

Function ζ is determined with an accuracy to within the constant and the integral $x\ (\theta) \to \infty$ when $\theta \to \theta^-$ (for the considered class of functions $f\ (\theta)$ whose properties are defined by (1.3) and $f\ (\theta_1) \neq 0$) which follows from the expansion of $x\ (\theta)$ in Taylor series in the neighborhood of point $\theta = \theta^-$. Hence by selecting θ_2

as the flow-on trajectory $\theta_2 = \theta^-$ and setting $F(\infty) = 0$ we obtain the following system of boundary conditions for Eq. (1.7):

$$c(\eta, 0) = 1, \quad c(0, \zeta) = 0, \quad c(\infty, \zeta) = 1 \tag{1.11}$$

The solution of problem (1.7), (1.11) is of the form

$$c(\eta, \zeta) = \operatorname{erf}\left(\frac{1}{2}\eta\zeta^{-1/2}\right) = \frac{2}{\sqrt{\pi}} \int\limits_0^{1/2\eta\zeta^{-1/2}} \exp(-u^2)\, du \tag{1.12}$$

Initial condition for the input problem (1.1), (1.2), as implied by (1.9), is given by formula (1.12) where

$$\zeta(0, \theta) = \zeta_0 = -\int\limits_{\theta^-}^{\theta} f(\xi)\sin\xi\,\Omega^{-1}[t(\xi, x(\xi))]\, d\xi + F[x(\theta)] \tag{1.13}$$

For the diffusion boundary layer thickness δ, and for the differential and total diffusion fluxes on the drop surface from (1.4) and (1.12) we obtain

$$\delta = \frac{1}{j}, \quad j(t, \theta) = \frac{\partial c}{\partial y}\bigg|_{y=0} = \sqrt{\frac{P}{\pi}}\,\frac{f(\theta)}{\sqrt{\zeta(t, \theta)}} \tag{1.14}$$

$$J(t) = \int\limits_{\sigma} j\, d\sigma = 2\pi \int\limits_{\theta^+}^{\theta^-} j(t, \xi)\sin\xi\, d\xi$$

Since $f(\theta^+) = 0$ and $\zeta(t, \theta^+) \neq 0$ (see (1.3), (1.9), and (1.10) where $\theta_2 = \theta^-$), the diffusion boundary layer thickness δ becomes infinitely great at the flow-off point: $\delta \to \infty$, and $\theta \to \theta^+$.

When the flow is stabilized $\Omega(t) = \Omega^{(0)} = \text{const}$ and the assumption $F(\infty) = 0$ is valid, the concentration distribution and the diffusion flux on the drop surface settle, as implied by (1.9), (1.10), (1.12), and (1.14), in the steady mode [1]

$$t \to \infty, \quad c \to \operatorname{erf}[{}^1/{}_2\eta\zeta^{-1/2}(\infty, \theta)] \tag{1.15}$$

$$j(\infty, \theta) = \sqrt{\frac{P}{\pi}}\, f(\theta)\,\zeta^{-1/2}(\infty, \theta), \quad \zeta(\infty, \theta) = -\frac{1}{\Omega^{(0)}}\int\limits_{\theta^-}^{\theta} f(\xi)\sin\xi\, d\xi$$

when $t \to \infty$.

Further on we shall seek the explicit form of the solution of problem (1.1), (1.2) with two different initial conditions (see Sect. 2)

$$t = 0, \quad c = 1, \quad \zeta_0 = 0 \quad \left(F_\alpha(x) = \int\limits_{\theta^-}^{\theta(x)} f(\xi)\sin\xi\,\Omega^{-1}[t(\xi, x)]\, d\xi\right) \tag{1.16}$$

$$t = 0, \quad c = \operatorname{erf}\left[\frac{1}{2}\eta\zeta_0^{-1/2}\right], \quad \zeta_0 = -\int\limits_{\theta^-}^{\theta} f(\xi)\sin\xi\, d\xi \tag{1.17}$$

$$(F_\beta(x) = F_\alpha(x) + \zeta_0)$$

Function $\theta = \theta(x)$ is obtained by the inversion of function $x = x(\theta)$, defined in (1.9). In these formulas and in what follows the subscripts α and β denote quantities related to the first (1.16) and second (1.17) initial conditions, respectively. As seen from (1.12) and (1.13), the first initial condition implies that when $t < 0$ the concentration in the stream is initially constant and then suddenly the reaction begins to take place at the drop surface (a similar problem was considered in [2] for the case of stabilized potential flow around a spherical drop of a translational stream of perfect fluid). The second condition corresponds to the state when the substance concentration distribution in the fluid at the initial instant of time is determined by a steady diffusion mode with $\Omega^{(0)} = 1$.

2. Convective diffusion to a bubble in the case of uniformly translational and shear streams. We shall derive the solution of problem (1.1), (1.2) with initial conditions (1.16) and (1.17).

Let us consider the unsteady diffision to a bubble in an unsteady, uniformly translational and shear streams. We assume the following time dependence:

$$\Omega(t) = (1 + wt)^{-1} \tag{2.1}$$

Function $\Omega(t)$ may be represented as

$$\Omega(t) = 1 + \Omega^{(1)}(t), \quad \Omega^{(1)}(t) = -wt(1 + wt)^{-1}$$

Such velocity field near the surface is the result of superposition of the unsteady stream $\psi^{(1)} = {}^1/_2\Omega^{(1)}(t)\, y \sin^2\theta$ on the steady stream $\psi^{(0)} = {}^1/_2 y \sin^2\theta$ [4]. At small times function $\Omega^{(1)}(t) \to -wt$, and when $t \to \infty$ it reaches the limit value equal minus unity.

In what follows, the solution of problem (1.1), (1.2), (2.1) may be used for analyzing unsteady diffusion in a stabilized flow $(w \to 0)$ past a bubble. Moreover, it will be shown in Sect. 4 that several terms of the expansion of that solution when t is small (up to the first term containing w inclusively) provide the asymptotics of solution of the problem of the uniformly accelerated flow field.

Translational stream. For such stream the stream function is determined by formulas (1.2) and (2.1) where $f(\theta) = {}^1/_2 \sin^2\theta$, and angles $\theta^+ = 0$ and $\theta^- = \pi$ define, the flow-off and flow-on trajectories, respectively [4]. Using the results obtained in Sect. 1, we obtain the following first general solutions of system (1.8):

$$\Omega(t)\,[(1 + \cos\theta)/(1 - \cos\theta)]^w = C_1 \tag{2.2}$$

$$\zeta = -(2C_1)^{-1}\int_\pi^\theta \sin^3\xi \,\mathrm{ctg}^{2w}\left(\frac{\xi}{2}\right) d\xi + C_2$$

From this we obtain the expressions for variables ζ, and η with the first (1.16) and second (1.17) initial conditions, respectively,

$$\begin{aligned}\zeta_\alpha = 4\Omega^{-1}(t)\,\mathrm{tg}^{2w}(\theta/2)\,[&B(\cos^2(\theta/2), 2 + w, 2 - w) - \\ &B(z(t,\theta), 2 + w, 2 - w)]\end{aligned} \tag{2.3}$$

$$\zeta_\beta = \zeta_0^* + \zeta_\alpha, \quad \eta_\alpha = \eta_\beta = {}^1/_2 P^{1/2} y \sin^2\theta$$

$$\zeta_0^* = 2z^2 - {}^4/_3 z^3, \quad \zeta_0^*|_{t=0} = \zeta_0$$

$$z = z(t,\theta) = [1 + \Omega^{-1/w}(t)\,\mathrm{tg}^2(\theta/2)]^{-1}, \quad B(\chi, p, q) = \int_0^\chi \xi^{p-1}(1 - \xi)^{q-1} d\xi$$

where $B(\chi, p, q)$ is the incomplete beta function, and the diffusion boundary layer thickness and the diffusion fluxes on the bubble surface are determined by formulas (1.14) and (2.3).

When $w > 0$ and $t \to \infty$ we have for ζ the following asymptotic representation:

$$t \to \infty, \quad \zeta_{\alpha,\beta} \to 4\Omega^{-1}(t)\operatorname{tg}^{2w}(\theta/2)B\,(\cos^2(\theta/2),\ 2+w,\ 2-w) \qquad (2.4)$$

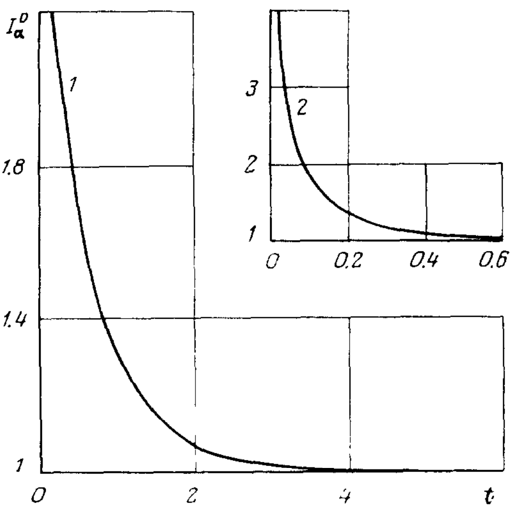

Fig. 1

Hence in the case of considerable times the total and local diffusion fluxes (1.14) assume the same mode and tend to zero in inverse proportion to $\sqrt{t}$. The latter means that the diffusion boundary layer thickness δ increases in proportion to $\sqrt{t}$, i.e. the diffusion boundary layer approximation (1.1) becomes invalid when the time is fairly long, and it is necessary to consider the process of diffusion without allowance for convective transfer. This is also clear if one takes into account that in the case of considerable time the bubble velocity is low, which corresponds to low Péclet numbers.

Let us now consider the case of the stabilized flow. For this we direct in formula (2.3) w to zero. Note that in the considered case the problem of convective diffusion with initial condition (1.17) reduces to that of conventional steady diffusion [1 – 3]. Hence problem (1.1) with initial condition (1.16) is of interest, since it is there that unsteady diffusion occurs under steady flow conditions.

Taking into account that $\Omega^{-1/w}(t) \to e^t$ when $w \to 0$, from (2.3) we obtain

$$\zeta_\alpha(t,\theta) = {}^1/_2[\cos\theta - {}^1/_3\cos^3\theta + {}^1/_3\,(V-1)^3(V+1)^{-3} - (V-1)\cdot(V+1)^{-1}] \qquad (2.5)$$

$$V = V(t,\theta) = e^{-t}\operatorname{ctg}^2(\theta/2)$$

From this, using the substitution $\lambda = \cos\theta$, for the total diffusion flux on the bubble surface we obtain the expression

$$I_\alpha(t) = \sqrt{6\pi P}\int_{-1}^{1}\frac{(1-\lambda^2)\,d\lambda}{\sqrt{3\lambda-\lambda^3+H^3-3H}} \qquad (2.6)$$

$$H = H(t,\lambda) = \frac{\lambda - \operatorname{th}(t/2)}{1-\lambda\operatorname{th}(t/2)}$$

$$I_\alpha \to {}^4/_3\sqrt{6\pi P}, \quad t \to \infty$$

The functional dependence $I_\alpha^0 = I_\alpha(t)/I_\alpha(\infty)$ is shown in Fig. 1 by curve 1.

It will be seen that the total flux on the bubble surface rapidly (exponentially) attains the steady mode ($t \to \infty$, [3]).

S h e a r s t r e a m. In this case the stream function near the bubble surface is defined by formulas (1.2) and (2.1), where $f(\theta) = 3 \sin^2 \theta \cos \theta$, $\theta_1^+ = 0$, $\theta_2^+ = \pi$ are flow-off trajectories, and $\theta_1^- = \pi / 2$ is the flow-on trajectory [5].

The first general solutions of system (1.9) are

$$\frac{\Omega(t)}{\mathrm{tg}^\gamma \theta} = C_1, \quad \zeta = \left(\frac{3}{2}\right) C_1^{-1} \int\limits_0^{\cos^2\theta} \lambda^{\gamma/2} (1 - \lambda)^{1-\gamma/2} d\lambda + C_2, \quad \gamma = \frac{w}{3} \qquad (2.7)$$

(the substitution $\lambda = \cos^2 \theta$) was made in the last integral). Thickness δ and the diffusion fluxes j and I are determined by formulas (1.14) where variables ζ and η at the first (1.16) and second (1.17) initial conditions are, respectively,

$$\begin{aligned} \zeta_\alpha &= {}^3/_2\Omega^{-1}(t)\, \mathrm{tg}^\gamma\theta[\mathrm{B}(\cos^2\theta,\ 1 + \gamma/2, 2 - \gamma/2) - \\ &\quad \mathrm{B}(m\ (t, \theta), 1 + \gamma/2, 2 - \gamma/2)] \\ \zeta_\beta &= \zeta_0{}^* + \zeta_\alpha, \quad \eta_\alpha = \eta_\beta = 3P^{1/2}\, y \sin^2\theta \cos\theta \\ \zeta_0{}^* &= {}^3/_4[1 - \Omega^{-4/\gamma}(t)\, \mathrm{tg}^4\theta m^2(t, \theta)], \quad \zeta_0{}^*|_{t=0} = \zeta_0 \\ m(t, \theta) &= [1 + \mathrm{tg}^2\theta\Omega^{-2/\gamma}(t)]^{-1} \end{aligned} \qquad (2.8)$$

which for $w > 0$ $(\gamma > 0)$ and considerable times yield for ζ the asymptotic expression

$$t \to \infty, \quad \zeta_{\alpha,\beta} \to {}^3/_2\Omega^{-1}(t)\, \mathrm{tg}^\gamma\theta \mathrm{B}(\cos^2\theta,\ 1 + \gamma/2, 2 - \gamma/2) \qquad (2.9)$$

It is evident from (2.9) that in this case function ζ behaves similarly to the corresponding function ζ in the case of uniformly translational stream when $t \to \infty$, hence all of the reasoning related to the latter is also valid here.

Directing in (2.8) $w \to 0$ $(\gamma \to 0)$ we obtain the problem of convective diffusion under condition of stabilized flow. As in the case of translational stream with stabilized flow past the bubble, problem (1.1), (1.17) leads here to the solution that defines steady diffusion [6].

The solution of problem (1.1), (1.16) is obtained by passing in formula (2.8) to limit $w \to 0$ $(\gamma \to 0)$. For the variable ζ_α and the total diffusion flux we have

$$\begin{aligned} \zeta_\alpha &= -{}^3/_4[\sin^4\theta - (e^{-6t}\, \mathrm{ctg}^2\theta + 1)^{-2}] \\ I_\alpha &= 4\sqrt{3\pi P}\, \mathrm{cth}^{1/2}(3t); \quad I_\alpha \to 4\sqrt{3\pi P}, \ t \to \infty \end{aligned} \qquad (2.10)$$

The dependence $I_\alpha^\circ = I_\alpha(t) / I_\alpha(\infty)$ is shown in Fig. 1 by curve 2. In this case the total diffusion flux exponentially attains the steady mode.

3. The velocity field of fluid in an unstable flow past a bubble at low Reynolds numbers. As shown in Sect. 1 and 2, the solution of the problem of convective diffusion on the bubble surface in an unsteady flow requires the determination of the flow field near that surface by the given velocity distribution in the fluid away from it, i.e. to determine the specific form of function $\Omega(t)$ in (1.2).

We shall consider here the translational and shear streams at infinity. The first obtains when a bubble rises in a viscous fluid, and the second occurs, for instance, when the bubble moves in a divergent (or convergent) flow in a conical channel.

The equation and boundary conditions for the stream function $\psi = \psi^{(0)}(r, \theta) + \psi^{(1)}(\tau, r, \theta)$ ($\psi^{(0)}$ is the stream function corresponding to stabilized motion with $\tau \leqslant 0$) are as follows:

$$E^2\left(E^2 - \frac{\partial}{\partial \tau}\right)\psi_n^{(1)} = 0, \quad \psi_n^{(1)}(0, r, \theta) = 0 \quad (n = 1, 2) \tag{3.1}$$

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

$$\left(v_r^{(1)} = \frac{1}{r^2\sin\theta}\frac{\partial\psi^{(1)}}{\partial\theta}, \quad v_\theta^{(1)} = -\frac{1}{r\sin\theta}\frac{\partial\psi^{(1)}}{\partial r}\right)$$

where $n = 1$ relates to a uniformly translational stream and $n = 2$ to a shear one (here and in what follows the subscript n is omitted, except when this could result in confusion); $v_r^{(1)}$ and $v_\theta^{(1)}$ are velocity components of the fluid. The units of length, time, and velocity are selected as follows: the bubble radius a, $a^2\nu^{-1}$ (ν is the kinematic viscosity of fluid), and $U = U_0(2 - n) + \alpha_0 a(n - 1)$ (U_0 is the stream velocity at infinity $(n = 1)$ and α is the coefficient of shear $(n = 2)$), respectively.

The boundary conditions which define the impermeability of the bubble boundary, absence in it of shearing stresses, and also the behavior of the stream function at infinity are of the form

$$r = 1, \quad v_r^{(1)} = 0, \quad \frac{\partial}{\partial r}\left(\frac{1}{r}v_\theta^{(1)}\right) + \frac{\partial}{\partial\theta}v_r^{(1)} = 0 \tag{3.2}$$

$$r \to \infty, \quad \psi^{(1)} \to {}^1/_2 n u^{(1)}(\tau) r^{n+1}\sin^2\theta\cos^{n-1}\theta \quad (u^{(1)}(0) = 0)$$

We seek a solution of problem (3.1), (3.2) of the form

$$\psi^{(1)} = \Phi(r, \tau)\sin^2\theta\cos^{n-1}\theta$$

Applying to (3.1) the Laplace transformation we obtain the following equation for images:

$$\left[\frac{d^2}{dr^2} - \frac{2(2n-1)}{r^2}\right]\left[\frac{d^2}{dr^2} - \frac{2(2n-1)}{r^2} - s\right]\Phi^* = 0$$

$$\Phi^*(r, s) = \int_0^\infty e^{-s\tau}\Phi(r, \tau)\,d\tau$$

Using the commutivity of operators in the left-hand side of this equation we obtain its general solution

$$\Phi^* = A_1 r^{n-1} + A_2 r^{-n} + r^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}[A_3\exp(s^{1/2}r) + A_4\exp(-s^{1/2}r)]$$

We determine constants A_j using boundary conditions (3.2) and obtain the following expressions for the unsteady addition to the stream function:

$$\psi^{(1)} = \sin^2\theta\cos^{n-1}\theta\,\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{s\tau}\Phi^*(r, s)\,ds \tag{3.3}$$

$$\Phi^* = u^*(s)\{r^{n+1} - [1 + 2s^{-1}\varkappa_n(1, s)]r^{-n} + 2s^{-1}\varkappa_n(r, s)r^{-n}\exp[s^{1/2}(1 - r)]\}$$

$$\varkappa_1(r, s) = 3(1 + s^{1/2}r)/(3 + s^{1/2}), \quad \varkappa_2(r, s) = 5(3 + 3s^{1/2}r + sr^2)/(5 + 5s^{1/2} + s)$$

where $u^*(s)$ are images of functions $u^{(1)}(\tau)$. Note that the expression for Φ^* in (3.3) in the case of translational stream $(n = 1)$ was obtained earlier (see, e. g., [7–9]).

Since in images the transition to limit $s \to \infty$ corresponds to $\tau \to 0$ in the originals, hence from formulas (3.3) we obtain

$$\tau \to 0, \quad \psi^{(1)} \to {}^1/_2 n u^{(1)}(\tau)(r^{n+1} - r^{-n}) \sin^2\theta \cos^{n-1}\theta \tag{3.4}$$

In what follows we shall need the velocity field in the fluid near the bubble surface. Expanding (3.3) in series in $y = r - 1$ we obtain

$$\begin{gathered}\psi^{(1)} = {}^1/_2 n K^{(1)}(\tau) y \sin^2\theta \cos^{n-1}\theta + O(y^2) \\ K^{(1)}(\tau) = \frac{1}{2\pi i} \int\limits_{\omega - i\infty}^{\omega + i\infty} e^{s\tau} \varkappa(1, s) u^*(s)\, ds\end{gathered} \tag{3.5}$$

Since $\tau = t/R$ $(R = Ua/\nu$ is the Reynolds number), it follows from (3.5) that the stream function near the particle surface can be represented in both cases of uniformly translational $(n = 1)$ and shear $(n = 2)$ flows in the form (1.2) where $\Omega(t) = \Omega^{(0)} + \Omega^{(1)} = [2n - 1 + K^{(1)}(\tau)]n/2$.

Functions $\varkappa(1, s)$ have the following properties:

$$s \to 0, \quad \varkappa(1, s) \to 2n - 1; \quad s \to \infty, \quad \varkappa(1, s) \to 2n + 1$$

which means that when appropriate limits exist, the relationships

$$\begin{gathered}\tau \to 0, \quad K^{(1)}(\tau) \to (2n + 1) u^{(1)}(\tau) \\ \tau \to \infty, \quad K^{(1)}(\tau) \to (2n - 1) u^{(1)}(\tau)\end{gathered} \tag{3.6}$$

are satisfied. Note that the first of these may be also obtained from (3.4) by expanding it in series in the small quantity $y = r - 1$.

Let us consider a specific example of the determination of function $K^{(1)}(\tau)$. For this we take functions $u^{(1)}(\tau)$ of the form

$$u_1^{(1)}(\tau) = \tau + {}^2/_3 \sqrt{\tau/\pi}, \quad u_2^{(1)}(\tau) = {}^1/_3 \sqrt{\tau/\pi}\,(20\tau + 15\sqrt{\pi\tau} + 6)$$

from which, using (3.5), we obtain

$$K_1^{(1)}(\tau) = \tau + 2\sqrt{\tau/\pi}, \quad K_2^{(1)}(\tau) = 5\sqrt{\tau/\pi}\,(4\tau + 3\sqrt{\pi\tau} + 2)$$

Thus for determining velocity fields near the bubble surface (i.e. for determining function $\Omega(t)$) conforming to a given flow away from a particle it is necessary to determine function $K^{(1)}(\tau)$ in formula (3.5), i.e. to perform the inversion of the Laplace transformation of known functions. Then, taking into consideration that $\tau = t/R$, to pass to formula (1.2) and solve the related problem of diffusion (Sects. 1 and 2). In the case of arbitrary times it is necessary to resort to numerical methods, and when $t \to 0$ it is possible to obtain an analytic solution (see Sect. 4 below).

4 The asymptotic behavior of solutions in the case of short times. Let us analyze the asymptotic behavior of solutions of problem (1.1) with initial conditions (1.16) and (1.17) and $t \to 0$. For the translational flow past the bubble by directing t to zero in (1.14) and (2.3) we obtain for total diffusion fluxes the following asymptotic expressions:

$$t \to 0, \quad I_\alpha(t) = 4\sqrt{\pi P/t}\,[1 + (5/72)t^2 - (7/72)wt^3 + O(t^4)] \qquad (4.1)$$
$$I_\beta(t) = {}^4/_3\sqrt{6\pi P}\,[1 - {}^3/_5(2\sqrt{3} - 3)wt^2 + O(t^3)]$$

By directing t to zero in (1.14) and (2.8) in the case of shear flow we obtain for total fluxes the respective asymptotic formulas

$$t \to 0, \quad I_\alpha(t) = 4\sqrt{\pi P/t}\,[1 + {}^3/_2t^2 - {}^6/_7(81 + 7w)t^3 + O(t^4)] \qquad (4.2)$$
$$I_\beta(t) = 4\sqrt{3\pi P}\,[1 - {}^3/_4(3\pi - 8)wt^2 + O(t^3)]$$

Formulas (4.1) and (4.2) imply that in the case of the first initial condition the total diffusion flux approaches infinity when $t \to 0$. This is due to the mismatching of the initial $c(0, 0, \theta) = 0$ and boundary $c(0, y \to +0, \theta) = 1$ conditions in (1.16) and (1.1).

Formulas for total diffusion fluxes (4.1) and (4.2) remain valid when $t \to 0$ for any $K(\tau)$ defined as follows:

$$\psi = {}^1/_2\, nK(\tau)\, y \sin^2\theta \cos^{n-1}\theta, \quad K(\tau) = 2n - 1 + K^{(1)}(\tau)$$
$$K^{(1)}(0) = 0; \quad \partial K / \partial t \to -(2n-1)w, \; t \to 0, \quad (\tau = R^{-1}t)$$

which is accurate to the first term with coefficient w in the expansion in powers of t. Taking this property into account the coefficients of transition (3.6) we shall investigate the convective diffusion on the surface of a bubble subjected to a uniformly accelerated motion defined by

$$r \to \infty, \quad \psi \to {}^1/_2 nu(\tau) r^{n+1} \sin^2\theta \cos^{n-1}\theta$$
$$u(\tau) = 1 + u^{(1)}(\tau) = 1 + \frac{2n-1}{2n+1} b\tau$$

where the coefficient at $b\tau$ is chosen for convenience. At small τ to this formula correspond the following functions $K(\tau)$ and $\Omega(t)$:

$$K(\tau) = (2n - 1)(1 + b\tau) \quad (\tau = R^{-1}t) \qquad (4.3)$$
$$\Omega(t) = 1 + bR^{-1}t$$

It follows from relationships (4.3) that $w = -bR^{-1}$. Hence using formulas (4.1) and (4.2) we obtain for total diffusion fluxes in the cases of translational ($n = 1$) and shear ($n = 2$) flows the following expressions:

$$I_\beta(t) = I_\beta(0)\,[1 + M_n bR^{-1}t^2 + o(t^2)] \qquad (4.4)$$
$$M_1 = {}^3/_5(2\sqrt{3} - 3), \quad M_2 = {}^3/_4(3\pi - 8)$$

It will be seen that formulas (4.4) do not contain the first term of expansion I_β in series in t. The second term contains $1/R$ as a factor. The contribution of this factor to the total diffusion flux, on assumptions made above $(R \ll 1)$, is significant. Note that the total flux increases with increase of speed of the bubble and diminishes with its deceleration.

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